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Bi-Relational Frameworks for Minimal and Intuitionistic Logics

by

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Kripke-type models for minimal logic have been proposed by Segerberg [6], Fitting [1] and Luckhardt [3]. These models are apparently motivated by the fact that $\perp \supset A$ does not necessarily constitute a thesis of minimal logic. ($\perp$ denotes a propositional constant which is contradictory.) In fact their models are obtained from those for intuitionistic logic by introducing a hereditary set of worlds with respect to accessibility relation, in which $\perp$ is made true. (The definition of hereditary sets will be given presently.) This is one of the characteristics of these models for minimal logic.

It is the purpose of the present paper to propose another kind of frames (model structures) for minimal and intuitionistic logics, which are obtained from the frames $\langle W,R\rangle$ for intuitionistic logic by adjoining thereto an additional relation $Q \subseteq R$ to be specified below. By such bi-relational frames we are naturally led to some bi-modal logics, which amount to four in number, and two of them correspond to minimal logic with the other two corresponding to intuitionistic logic.

We note in ~~passing that~~ $\neg$ is not defined, but constitutes a primitive logical symbol along with conjunction ($\wedge$), disjunction ($\vee$) and implication ($\supset$) and we will not employ $\perp$, therefore.

Before proceeding we wish to briefly recapitulate with some inessential modifications the completeness result as obtained by Segerberg [6] and some preliminaries to be assumed.

Definition: A frame for minimal logic is an ordered triple $\langle W,R,N\rangle$ such that

a) W is a non-empty set (of possible worlds),

b) R is a binary relation, reflexive and transitive, defined on W,

c) N is a hereditary subset of W with respect to R i.e. a subset of W such that

$$\forall\Gamma\Delta(\Gamma R\Delta \text{ and } \Gamma\in N \Rightarrow \Delta\in N)$$

where $\Gamma,\Delta,\ldots$ are meta-logical variables ranging over the elements of W.

In terms of the frame $\langle W,R,N\rangle$ thus defined a model for minimal logic is then defined as an ordered quadruple $\langle W,R,N,\models\rangle$ such that $\models$ is a relation between elements of W and formulas satisfying for every Γ and every A and B:

d) $\forall\Delta(\Gamma R\Delta$ and $\Gamma\models A \Rightarrow \Delta\models A)$

e) $\Gamma\models A\wedge B \iff \Gamma\models A$ and $\Gamma\models B$

f) $\Gamma\models A\vee B \iff \Gamma\models A$ or $\Gamma\models B$

g) $\Gamma\models A\supset B \iff \forall\Delta(\Gamma R\Delta \Rightarrow \Delta\not\models A$ or $\Delta\models B)$

h) $\Gamma\models\neg A \iff \forall\Delta(\Delta\notin N$ and $\Gamma R\Delta \Rightarrow \Delta\not\models A)$.

The validity of formulas is defined in the usual way and the completeness result follows:

Minimal (intuitionistic) logic is determined by M (I)

where M is the collection of all the frames for minimal logic and I the sub-collection thereof satisfying $N = \emptyset$.

1. *Bi-relational Frameworks*

Definition: A bi-relational frame for minimal logic is an ordered triple $\langle W,R,Q\rangle$ such that

a) W is a non-empty set (of possible worlds),

b) R is a binary relation, reflexive and transitive, defined on W,

c) Q is a sub-relation of R satisfying

c_1 $\quad \forall\Gamma\Delta\Lambda(\Gamma R\Delta \text{ and } \Delta Q\Lambda \Rightarrow \Gamma Q\Lambda)$

c_2 $\quad \forall\Gamma\Delta(\Gamma Q\Delta \Rightarrow \exists\Lambda(\Delta Q\Lambda))$.

In terms of a bi-relational frame $\langle W,R,Q\rangle$ a model for minimal logic is defined as an ordered quadruple $\langle W,R,Q,\models\rangle$ such that

d) - g) of the preceding definition and

h') $\Gamma \models \neg A \Leftrightarrow \forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \not\models A)$.

In this definition and the preceding one we could weaken the condition d) so that the A was an atomic formula and prove d) by induction on the degree of A.

A Kripke-type frame for intuitionistic logic is identical with a bi-relational frame $\langle W,R,Q\rangle$ for minimal logic with $R = Q$. It will be of use later on to observe that $R = Q$ is equivalent to the reflexivity of Q. The necessity is straightforward. For proving the converse suppose the reflexivity of Q and $\Gamma R\Delta$, which in conjunction give rise to $\Gamma Q\Delta$ by c_1.

In a bi-relational frame $\langle W,R,Q\rangle$ Q can be restricted by the following additional condition, which stops short of making it reflexive, but still continues to provide a frame for intuitionistic logic:

c_3 $\qquad \forall\Gamma\,\exists\Delta(\Gamma Q\Delta).$

The condition is obviously satisfied by a reflexive Q. (The reflexivity of Q will be refered to as c_3' in the sequel.)

On the other hand the condition c_2 is strengthened to:

c_2' $\qquad \forall\Gamma\Delta(\Gamma Q\Delta \Rightarrow \Delta Q\Delta).$

As will be seen the class of bi-relational frames for minimal logic thus strengthened is still sufficient to determine minimal logic.

The collections of bi-relational frames for minimal (intuitionistic) logic respectively satisfying c_2 and c_2' (c_3 and c_3') will be called *BM* and *BM'* (*BI* and *BI'*).

Obviously:

$$\begin{tikzcd}
 & BI \arrow[dr, phantom, sloped, "\subseteq"] & \\
BI' \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & BM \\
 & BM'. \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

THEOREM 1 *Minimal (intuitionistic) logic is determined by* *BM* *as well as by* *BM'* *(by* *BI* *as well as by* *BI').*

The theorem is obtained immediately from the following two lemmas.

LEMMA 2 *For every model* $\langle W,R,N,\vDash\rangle$ *for minimal (intuitionistic) logic there is a bi-relational model* $\langle W,R,Q,\vDash'\rangle$ *such that* $\langle W,R,Q\rangle \in$ *BM'* *(BI')* *satisfying that* $\Gamma \vDash A$ *iff* $\Gamma \vDash' A$ *for every* Γ *and every* A.

Proof. Let $Q = R \cap W\times(W-N)$.

Now, $\Gamma R\Delta$ and $\Delta Q\Lambda$

$\Rightarrow$ $\Gamma R\Delta$ and $\Delta R\Lambda$ and $\Lambda \notin N$

$\Rightarrow$ $\Gamma R\Lambda$ and $\Lambda \notin N$ (by the transitivity of R)

$\Rightarrow$ $\Gamma Q\Lambda$.

$\Gamma Q\Delta$

$\Rightarrow$ $\Delta R\Delta$ and $\Delta \notin N$ (by the reflexivity of R)

$\Rightarrow$ $\Delta Q\Delta$.

(Trivially $Q = R$ for $N = \emptyset$.)

This takes care of c_1 and c_2' (c_1 and c_3'), namely, $\langle W,R,Q\rangle \in$ *BM'* (*BI'*).

We next define $\vDash'$ for every Γ and every atomic A as $\Gamma \vDash' A$ if $\Gamma \vDash A$, then extend it to all formulas so that $\langle W,R,Q,\vDash'\rangle$ is a bi-relational model for minimal (intuitionistic) logic.

It is then proved by induction on the degree of A that for every A $\Gamma \models A$ iff $\Gamma \models' A$ as illustrated by:

$$\begin{aligned}
\Gamma \models \neg B &\Leftrightarrow \forall\Delta(\Delta \notin N \text{ and } \Gamma R \Delta \Rightarrow \Delta \not\models B) \\
&\Leftrightarrow \forall\Delta(\Gamma Q \Delta \Rightarrow \Delta \not\models B) \\
&\Leftrightarrow \forall\Delta(\Gamma Q \Delta \Rightarrow \Delta \not\models' B) \\
&\Leftrightarrow \Gamma \models' \neg B.
\end{aligned}$$

LEMMA 3 *For every bi-relational model* $\langle W,R,Q,\models\rangle$ *for minimal (intuitionistic) logic such that* $\langle W,R,Q\rangle \in BM$ *(BI) there is a model* $\langle W,R,N,\models'\rangle$ *such that* $\langle W,R,N\rangle \in M$ *(I) satisfying that* $\Gamma \models A$ *iff* $\Gamma \models' A$ *for every* Γ *and every* A.

Proof. Let N = W-Domain(Q) (= $\{\Gamma : \forall\Delta(\text{not } \Gamma Q \Delta)\}$).

N is hereditary with respect to R.

Since: $\Gamma R \Delta$ and $\Gamma \in N \Rightarrow \forall\Lambda(\text{not } \Gamma Q \Lambda)$

$\Rightarrow \forall\Lambda(\text{not } \Delta Q \Lambda)$ (by c_1)

$\Rightarrow \Delta \in N$.

(If for every Γ there is some Δ such that $\Gamma Q \Delta$, then N is certainly empty. Namely, $\langle W,R,N\rangle \in I$.)

Define $\Gamma \models' A$ for every Γ and every atomic A if $\Gamma \models A$ and extend it so that $\langle W,R,N,\models'\rangle$ becomes a model for minimal (intuitionistic) logic.

It is then proved by induction on the degree that $\Gamma \models A$ iff $\Gamma \models' A$ for every Γ and every A.

We illustrate the proof by the case for $A = \neg B$.

$\forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \nvDash B)$

$\Rightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \exists\Theta(\Lambda Q\Theta))$ (by the definition of N)

$\Rightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \exists\Theta(\Lambda Q\Theta$ and $\Gamma Q\Theta))$ (by c_1)

$\Rightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \exists\Theta(\Lambda R\Theta$ and $\Theta \nvDash B))$ (by $Q \subseteq R$ and assumption)

$\Rightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \Lambda \nvDash B)$ (by d))

$\Rightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \Lambda \nvDash' B)$ (by induction hypothesis).

Conversely:

$\forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \Lambda \nvDash' B)$

$\Rightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \exists\Theta(\Delta Q\Theta))$ (by c_2)

$\Rightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \in \mathrm{Domain}(Q))$

$\Rightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \Gamma R\Delta$ and $\Delta \notin N)$ (by $Q \subseteq R$ and the definition of N)

$\Rightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \nvDash' B)$ (by assumption)

$\Rightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \nvDash B)$ (by induction hypothesis).

Thus,

$\Gamma \vDash \neg B \quad \Leftrightarrow \quad \forall\Delta(\Gamma Q\Delta \Rightarrow \Delta \nvDash B)$

$\Leftrightarrow \quad \forall\Lambda(\Lambda \notin N$ and $\Gamma R\Lambda \Rightarrow \Lambda \nvDash' B)$

$\Leftrightarrow \quad \Gamma \vDash' \neg B.$

On the basis of these results we next prove a theorem, which describes a relationship between minimal and intuitionistic logics.

LEMMA 4 *If* A *does not contain* $\supset$, *then*

$$I \vdash \neg A \iff M \vdash \neg A$$

where I *and* M *denote respectively intuitionistic and minimal logics.*

Proof. $\Leftarrow$ is obvious. For proving $\Rightarrow$ assume $M \nvdash \neg A$. There is then a model $\langle W,R,Q,\models\rangle$ for minimal logic with $\langle W,R,Q\rangle \in BM$ and a Γ such that $\Gamma \nvDash \neg A$. By h') this gives $Q \neq \emptyset$. Putting $W' = \text{Domain}(Q)$ and $R' = R \cap W' \times W'$, we have $Q \subseteq W' \times W'$ by c_2, from which follows $Q \subseteq R'$. We therefore have $\langle W',R',Q\rangle \in BI$.

For every $\Delta \in W'$ and every atomic A, $\Delta \models' A$ is defined as $\Delta \models A$ obtaining a bi-relational model $\langle W',R',Q,\models'\rangle$ for intuitionistic logic. By induction on the degree of B it is proved that $\Delta \models' B$ iff $\Delta \models B$ for every $\Delta \in W'$ and every B not containing $\supset$. Obviously $\Gamma \in W'$. Hence $\Gamma \nvDash' \neg A$ and $I \nvdash \neg A$ by the consistency of intuitionistic logic.

The lemma just proved is generalized as follows:

THEOREM 5 *If in* A *the scopes of* $\supset$ *and* $\neg$ *do not overlap with each other, then*

$$I \vdash A \iff M \vdash A.$$

Proof. $\Leftarrow$ is obvious. $\Rightarrow$ is again proved by induction on the degree of A.

(i) $I \vdash A \wedge B \Rightarrow I \vdash A$ and $I \vdash B$
$\Rightarrow M \vdash A$ and $M \vdash B$
$\Rightarrow M \vdash A \wedge B$.

(ii) $I \vdash A \vee B \Rightarrow I \vdash A$ or $I \vdash B$ (by disjunction property of intuitionistic logic)
$\Rightarrow M \vdash A$ or $M \vdash B$
$\Rightarrow M \vdash A \vee B$.

(iii) $I \vdash A \supset B \Rightarrow M \vdash A \supset B$ (since $A \supset B$ is already a thesis of positive logic).

(iv) $I \vdash \neg A \Rightarrow M \vdash \neg A$ (by Lemma 4, since A does not contain $\supset$).

As is well-known any formula containing only $\wedge$ and $\neg$ is provable in classical logic iff it is a thesis of intuitionistic logic (Gödel). From the above theorem follows then the generalization of this result to minimal logic, which was already mentioned in Schmidt [5].

2. *Embedding Theorems*

Four bi-modal logics are introduced for the purpose.

S4-D*4 is a bi-modal logic to be obtained from classical logic by adjoining the following classes of axioms:

$K_1 \qquad \Box_1(A\supset B)\supset(\Box_1 A\supset\Box_1 B)$

$T_1 \qquad \Box_1 A\supset A$

$4_1 \qquad \Box_1 A\supset\Box_1\Box_1 A$

$K_2 \qquad \Box_2(A\supset B)\supset(\Box_2 A\supset\Box_2 B)$

$D^*_2 \qquad \Box_2(\Box_2 A\supset\Diamond_2 A)$ (or equivalently $\Box_2\Diamond_2\top$)

$4'_2 \qquad \Box_2 A\supset\Box_1\Box_2 A$

$\Box_1\text{-}\Box_2 \qquad \Box_1 A\supset\Box_2 A$

where $\Diamond_1(\Diamond_2)$ stands for $\sim\Box_1\sim$ $(\sim\Box_2\sim)$ and $\top$ a constantly true formula such as $A\supset A$.

Rules: $\vdash A,\ \vdash A\supset B \Rightarrow \vdash B$ (detachment)

$\vdash A \Rightarrow \vdash \Box_1 A$ (necessitation for $\Box_1$).

Bi-modal logics S4-D4, S4-T*4 and S4-S4 are obtained from S4-D*4 as above defined by replacing D^*_2 respectively by:

$D_2 \qquad \Box_2 A\supset\Diamond_2 A$ (or equivalently $\Diamond_2\top$)

$T^*_2 \qquad \Box_2(\Box_2 A\supset A)$

$T_2 \qquad \Box_2 A\supset A.$

In S4-S4 we have $\Box_1 A\equiv\Box_2 A$, since $\Box_2 A\supset\Box_1 A$ by $4'_2$ and T_2, the converse being taken care of by $\Box_1\text{-}\Box_2$. S4-S4 is therefore a logic essentially the same as (uni-modal) S4.

These bi-modal logics are related to each other as illustrated by the diagram:

$$\begin{tikzcd}
\text{S4-S4} \arrow[r] \arrow[dr] & \text{S4-T*4} \arrow[dr] & \\
& \text{S4-D4} \arrow[r] & \text{S4-D*4.}
\end{tikzcd}$$

A model for bi-modal logic S4-D*4 (S4-D4, S4-T*4 and S4-S4) is a quadruple $\langle W,R,Q,\models_m\rangle$ such that $\langle W,R,Q\rangle \in$ *BM* (*BI*, *BM'* and *BI'*) satisfying:

$$\Gamma \models_m \Box_1 A \iff \forall\Delta(\Gamma R\Delta \implies \Delta \models_m A)$$

$$\Gamma \models_m \Box_2 A \iff \forall\Delta(\Gamma Q\Delta \implies \Delta \models_m A)$$

along with the usual requirements for the (classical) satisfying relation $\models_m$ for other logical symbols.

Each of these bi-modal logics is consistent and complete with respect to the corresponding semantics. For example,

$$\begin{aligned}\Gamma \models_m \Box_2 \Diamond_2 T &\iff \forall\Delta(\Gamma Q\Delta \implies \Delta \models_m \Diamond_2 T)\\ &\iff \forall\Delta(\Gamma Q\Delta \implies \exists\Lambda(\Delta Q\Lambda \text{ and } \Lambda \models_m T),\end{aligned}$$

which is obtained from c_2. D^*_2 i.e. $\Box_2 \Diamond_2 T$ is thus valid in *BM*.

$$\begin{aligned}&\Gamma \models_m \Box_2 A \supset \Box_1 \Box_2 A\\ \iff& \Gamma \not\models_m \Box_2 A \text{ or } \Gamma \models_m \Box_1 \Box_2 A\\ \iff& \exists\Delta(\Gamma Q\Delta \text{ and } \Delta \not\models_m A) \text{ or } \forall\Delta(\Gamma R\Delta \implies \Delta \models_m \Box_2 A)\\ \iff& \exists\Delta(\Gamma Q\Delta \text{ and } \Delta \not\models_m A) \text{ or } \forall\Delta\Lambda(\Gamma R\Delta \text{ and } \Delta Q\Lambda \implies \Lambda \models_m A),\end{aligned}$$

which is obtained by c_1. This proves the validity of $4'_2$ i.e. $\Box_2 A \supset \Box_1 \Box_2 A$ in *BM*.

The completeness is proved by the well-known method due to Lemmon-Scott [2].

With a view to proving embedding theorem we define a mapping σ, which transform every formula of intuitionistic logic into another belonging to bi-modal logic:

$$\sigma(A) = \Box_1 A \quad \text{for atomic } A$$
$$\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$$
$$\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$$
$$\sigma(A \supset B) = \Box_1(\sigma(A) \supset \sigma(B))$$
$$\sigma(\neg A) = \Box_2 {\sim} \sigma(A).$$

LEMMA 6 *For every bi-relational model* $\langle W,R,Q,\models\rangle$ *for minimal logic there is a model* $\langle W,R,Q,\models_m\rangle$ *for bi-modal logic such that* $\Gamma \models A$ *iff* $\Gamma \models_m \sigma(A)$ *for every* Γ *and every* A.

Proof. For every Γ and atomic A $\Gamma \models_m A$ is deffined as $\Gamma \models A$. We then prove by induction on the degree of A that $\Gamma \models A$ iff $\Gamma \models_m \sigma(A)$ for every Γ and every A.

The induction steps are illustrated by:

$$\begin{aligned}
\Gamma \models A \supset B &\iff \forall\Delta(\Gamma R \Delta \implies \Delta \not\models A \text{ or } \Delta \models B) \\
&\iff \forall\Delta(\Gamma R \Delta \implies \Delta \not\models_m \sigma(A) \text{ or } \Delta \models_m \sigma(B)) \\
&\iff \Gamma \models_m \Box_1(\sigma(A) \supset \sigma(B)) \\
&\iff \Gamma \models_m \sigma(A \supset B), \\
\Gamma \models \neg A &\iff \forall\Delta(\Gamma Q \Delta \implies \Delta \not\models A) \\
&\iff \forall\Delta(\Gamma Q \Delta \implies \Delta \not\models_m \sigma(A)) \\
&\iff \Gamma \models_m \Box_2 {\sim} \sigma(A) \\
&\iff \Gamma \models_m \sigma(\neg A).
\end{aligned}$$

We note in passing that the newly obtained model for bi-modal logic depends upon the frame of the given model. For example, a model for S4-D*4 will be obtained from a frame $\langle W,R,Q \rangle$ belonging to *BM*.

LEMMA 7 *For every model* $\langle W,R,Q,\models_m \rangle$ *for bi-modal logic there is a bi-relational model* $\langle W,R,Q,\models \rangle$ *for minimal logic such that* $\Gamma \models A$ *iff* $\Gamma \models_m \sigma(A)$ *for every* Γ *and every* A.

Proof. For every Γ and atomic A $\Gamma \models A$ is defined as $\Gamma \models_m \Box_1 A$. Then proceed by induction on the degree of A.

By Lemma 6 and 7 as well as Theorem 1 we obtain:

THEOREM 8

(i) $M \vdash A \iff$ S4-D*4 $\vdash \sigma(A) \iff$ S4-T*4 $\vdash \sigma(A)$

(ii) $I \vdash A \iff$ S4-D4 $\vdash \sigma(A) \iff$ S4-S4 $\vdash \sigma(A)$.

Since $\Box_1 A \equiv \Box_2 A$ in S4-S4, the second equivalence of (ii) is nothing but the celebrated embedding theorem of McKinsey-Tarski [4].

Appendix

The tableau proof system for intuitionistic logic by Fitting [1] consists of the following reduction rules:

$$T\wedge \quad \frac{S,\ TA\wedge B}{S,\ TA,\ TB} \qquad F\vee \quad \frac{S,\ FA\vee B}{S,\ FA,\ FB}$$

$$T\vee \quad \frac{S,\ TA\vee B}{S,\ TA \mid S,\ TB} \qquad F\wedge \quad \frac{S,\ FA\wedge B}{S,\ FA \mid S,\ FB}$$

$$T\supset \quad \frac{S,\ TA\supset B}{S,\ FA \mid S,\ TB} \qquad F\supset \quad \frac{S,\ FA\supset B}{S_T,\ TA,\ FB}$$

$$T\neg \quad \frac{S,\ T\neg A}{S,\ FA} \qquad F\neg \quad \frac{S,\ F\neg A}{S_T,\ TA}$$

where $S_T = \{ TA : TA \in S \}$. (For details consult Fitting [1].)

By restricting $T\neg$ to:

$$T\neg \quad \frac{S_T,\ T\neg A}{S_T,\ FA}$$

we obtain a tableau proof system for minimal logic.

By replacing $T\neg$ and $F\neg$ respectively by:

$$T\neg \quad \frac{S,\ T\neg A,\ F\neg B}{S_T,\ FA} \qquad F\neg \quad \frac{S,\ F\neg A}{S_T,\ TA,\ F\neg A}$$

we could also obtain a tableau proof system for minimal logic.

Employing any one of these tableau proof systems we could prove the same completeness result of minimal logic with respect to the proposed bi-relational frames in a more constructive way.

References

[1] Melvin Fitting. *Intuitionistic logic model theory and forcing.* North Holland, 1969.

[2] E.J.Lemmon and D.S.Scott. *Intensional logic: preliminary draft of initial chapters by E.J.Lemmon.* Mimeographed. Stanford, Calif.: Department of mathematics, Stanford University, July 1966.

[3] H.Luckhardt. "Kripke-Semantik der derivativen Prädikatelogik." *Archiv für mathematische Logik und Grundlagenforschung,* vol. 13 (1970), pp. 134-135.

[4] J.C.C.McKinsey and A.Tarski. "Some theorems about the sentential calculi of Lewis and Heyting." *The journal of symbolic logic,* vol. 13 (1948), pp. 1-15.

[5] H.A.Schmidt. *Mathematische Gesetze der Logik I.* Springer, 1960.

[6] Krister Segerberg. "Propositional logics related to Heyting's and Johansson's." *Theoria,* vol. 34 (1968), pp. 26-61.